Algorithms: DFS, STRONGLY CONNECTED COMPONENTS, FLOWS

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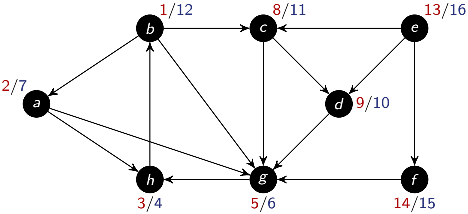
Pseudocode of DFS

```
DFS-VISIT($G, u$)
  $time = time + 1$
  $u.d = time$
  $u.color = $ GRAY                    // discover $u$
  for each $v \in G.Adj[u]$            // explore $(u, v)$
      if $v.color ==$ WHITE
          DFS-VISIT($v$)
  $u.color = $ BLACK
  $time = time + 1$
  $u.f = time$                         // finish $u$
```



time = 16

Runtime Analysis

```
DFS($G$)
  for each $u \in G.V$
      $u.color$ = WHITE
  $time = 0$
  for each $u \in G.V$
      if $u.color$ == WHITE
          DFS-VISIT($G, u$)
```

```
DFS-VISIT($G, u$)
  $time = time + 1$
  $u.d = time$
  $u.color$ = GRAY                        // discover $u$
  for each $v \in G.Adj[u]$               // explore $(u, v)$
      if $v.color$ == WHITE
          DFS-VISIT($v$)
  $u.color$ = BLACK
  $time = time + 1$
  $u.f = time$                            // finish $u$
```

- Color each vertex white takes time $\Theta(V)$
- Note that DFS-VISIT is called once for each vertex (when it is colored gray from white)
- When DFS-VISIT(u) is called the **for** loop runs at most $\{\#\text{neighbors of } u\}$ times
- Therefore the total time DFS-VISIT is run is

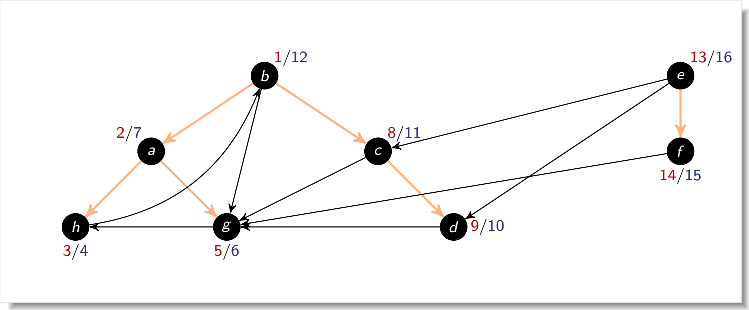
$$\sum_{u \in V} \{\#\text{neighbors of } u\} = \Theta(E)$$

Total Time: $\Theta(V + E)$

PROPERTIES OF DFS

Output of DFS

DFS forms a depth-first forest comprised of > 1 depth-first trees. Each tree is made of edges (u, v) such that u is gray and v is white when (u, v) is explored.



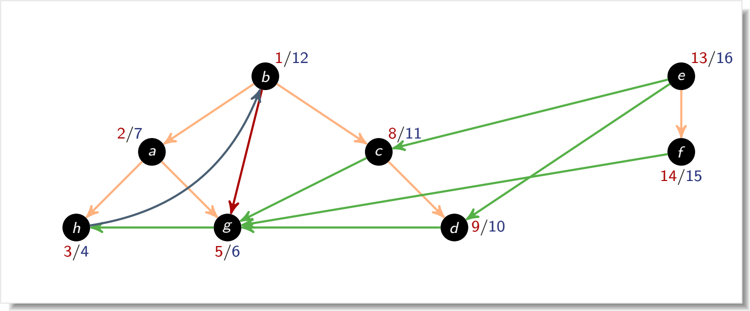
Classification of edges

Tree edge: In the depth-first forest, found by exploring (u, v)

Back edge: (u, v) where u is a descendant of v

Forward edge: (u, v) where v is a descendant of u, but not a tree edge

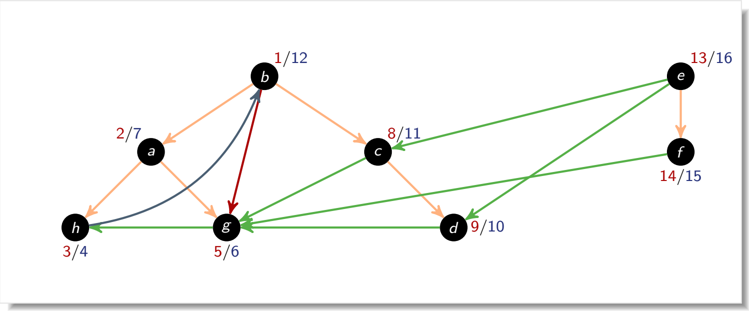
Cross edge: any other edge



Parenthesis theorem

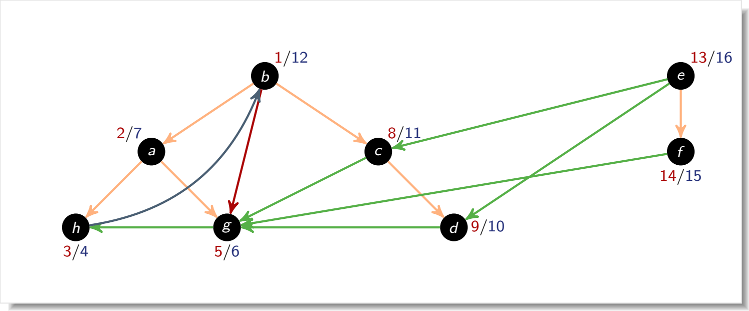
For all u, v exactly one of the following holds

1. $[u.d, u.f]$ and $[v.d, v.f]$ are disjoint neither of u and v are descendant of each other
2. $u.d < v.d < v.f < u.f$ and v is a descendant of u
3. $v.d < u.d < u.f < v.f$ and u is a descendant of v.



White-path theorem

Vertex v is a descendant of u if and only if at time $u.d$ there is a path from u to v consisting of only white vertices (except for u, which was just colored gray)



TOPOLOGICAL SORT

Application of DFS

Topological sort

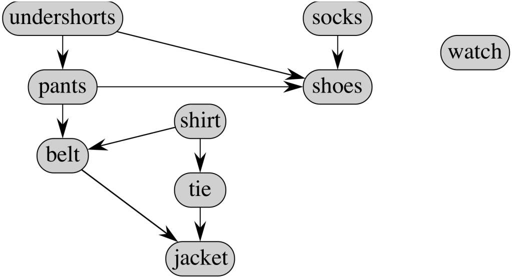
Definition

INPUT: A directed acyclic graph (DAG) $G = (V, E)$

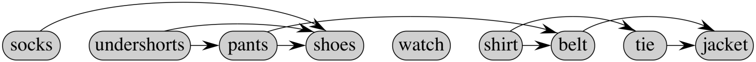
OUTPUT: a linear ordering of vertices such that if $(u, v) \in E$, then u appears somewhere before v

Example

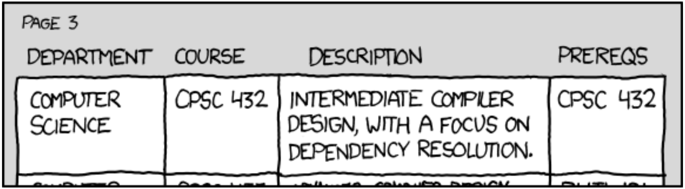
Getting dressed in the morning:



in which order?



First: when is a directed graph acyclic?



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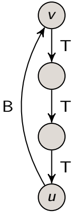
DEPARTMENT	COURSE	DESCRIPTION	PREREQS
COMPUTER SCIENCE	CPSC 432	INTERMEDIATE COMPILER DESIGN, WITH A FOCUS ON DEPENDENCY RESOLUTION.	CPSC 432

First: when is a directed graph acyclic?

Lemma

A directed graph G is acyclic if and only if a DFS of G yields no back edges

Proof. First show that back-edge implies cycle
Suppose there is a back edge (u, v). Then v is ancestor of u in depth-first forest. Therefore there is a path from v to u, which creates a cycle.

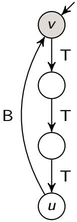


First: when is a directed graph acyclic?

Lemma

A directed graph G is acyclic if and only if a DFS of G yields no back edges

Proof. Second show that cycle implies back-edge
Let v be the first vertex discovered in the cycle C and let (u, v) be the preceding edge in C. At time $v.d$ vertices in C form a white-path from v to u and hence u is a descendant of v.

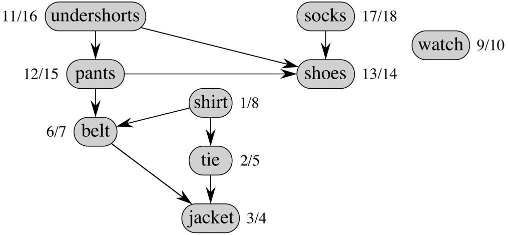


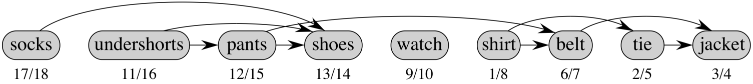
Algorithm for topological sort

TOPOLOGICAL-SORT(G):

1. Call $DFS(G)$ to compute finishing times $v.f$ for all $v \in G.V$
2. Output vertices in order of *decreasing* finishing times

Example





Time Analysis

```
TOPOLOGICAL-SORT($G$):
1. Call $DFS(G)$ to compute finishing times $v.f$ for all $v \in G.V$
2. Output vertices in order of decreasing finishing times
```

Do not need to sort by finishing times

- Can just output vertices as they are finished and understand that we want the reverse of the list
- Or put them onto the front of a linked list as they are finished. When done, the list contains vertices in topologically sorted order.

Time: $\Theta(V + E)$ (same as DFS)

Correctness

Need to show that if $(u, v) \in E$ then $v.f < u.f$

When we explore (u, v) what are the colors of u and v?

- u is gray
- Is v gray, too?
  - No, because then v would be ancestor of u which implies that there is a back edge so the graph is not acyclic (by previous Lemma)
- Is v white?
  - Then becomes descendant of u. By parenthesis theorem, $u.d < v.d < v.f < u.f$
- Is v black?
  - Then v is already finished. Since we are exploring (u, v), we have not yet finished u. Therefore, $v.f < u.f$.

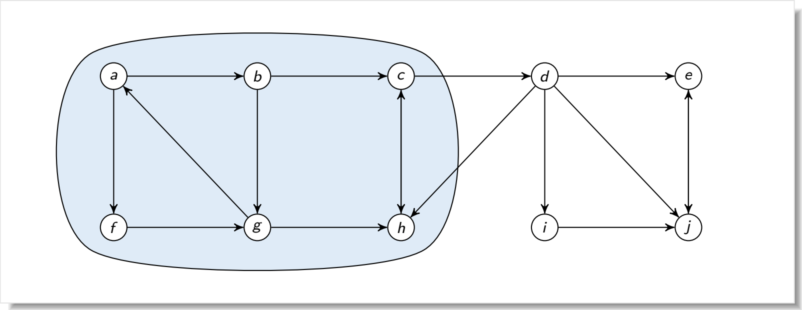
☺

STRONGLY CONNECTED COMPONENTS
(A magic algorithm)

What is a Strongly Connected Component?

Definition: A strongly connected component (SCC) of a directed graph $G = (V, E)$ is a maximal set of vertices $C \subseteq V$ such that for all $u, v \in C$, both $u \rightsquigarrow v$ and $v \rightsquigarrow u$.

Example:



Is this a SCC? NO, because e.g. $c \not\rightsquigarrow b$

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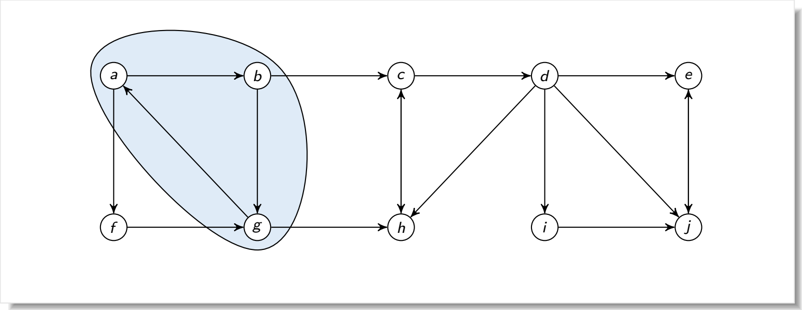

Is this a SCC? NO, because not maximal

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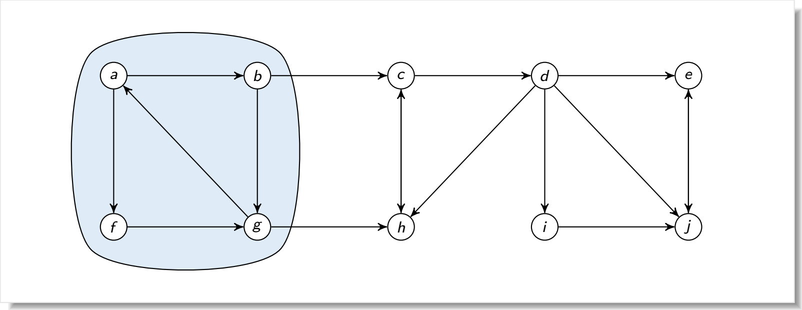
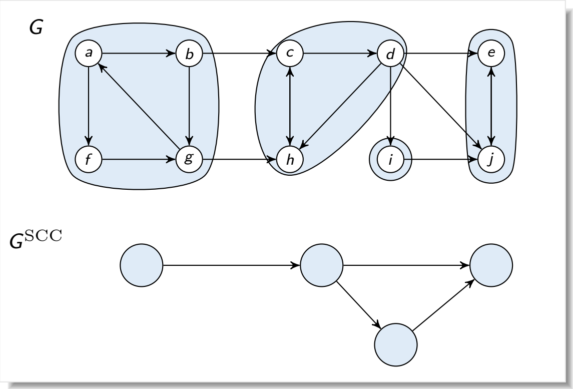


Is this a SCC? YES!

Component Graph

For a digraph $G = (V, E)$, its component graph $G^{\mathrm{SCC}} = (V^{\mathrm{SCC}}, E^{\mathrm{SCC}})$ is defined by:

- V^{SCC} has a vertex for each SCC in G;
- E^{SCC} has an edge if there's an edge between the corresponding SCC's in G.



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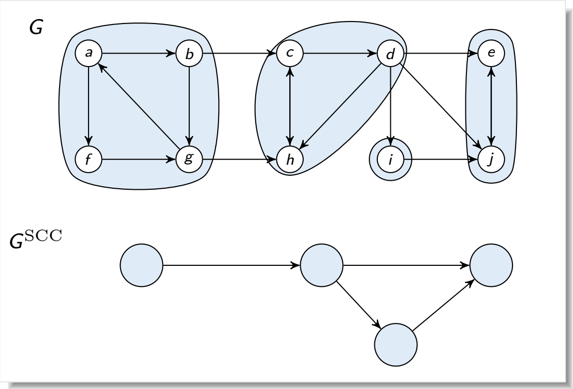

Lemma: G^{SCC} is a DAG.

Magic Algorithm

$\mathrm{SCC}(G)$:

1. Call $\mathrm{DFS}(G)$ to compute finishing times $u.f$ for all u.
2. Compute G^T
3. Call $\mathrm{DFS}(G^T)$ but in the main loop, consider vertices in order of decreasing $u.f$ (as computed in first DFS).
4. Output the vertices in each tree of the depth-first forest formed in second DFS as a separate SCC.

Graph G^T is the transpose of G:

- $G^T = (V, E), E^T = \{(u, v) : (v, u) \in E\}$.
- G^T is G with all edges reversed.

Observations:

- Can create G^T in $\Theta(V + E)$ time if using adjacency lists.
- G and G^T has the same SCCs.

Magic Algorithm

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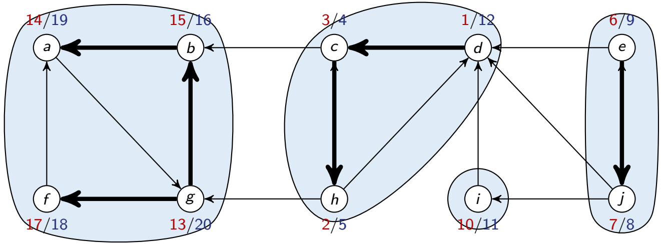


Analysis

SCC(G):

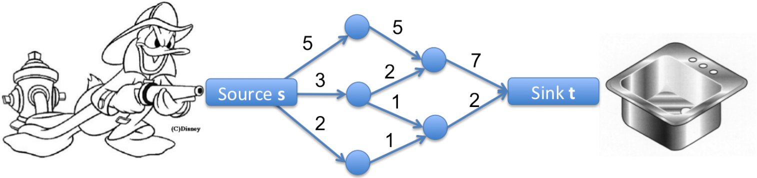
1. Call DFS(G) to compute finishing times $u.f$ for all u.
2. Compute G^T
3. Call DFS(G^T) but in the main loop, consider vertices in order of decreasing $u.f$ (as computed in first DFS).
4. Output the vertices in each tree of the depth-first forest formed in second DFS as a separate SCC.

Runtime analysis: Each step takes $\Theta(V + E)$ so total running time is $\Theta(V + E)$

Why does it work? Intuition:

- The first DFS orders SCC's in topological order (recall G^{SCC} is acyclic)
- Second DFS then outputs the vertices in each SCC

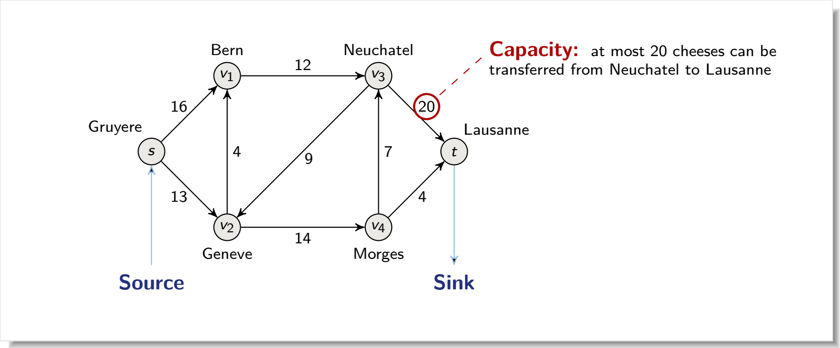
Formal proof in book



FLOW NETWORKS

Flow Network

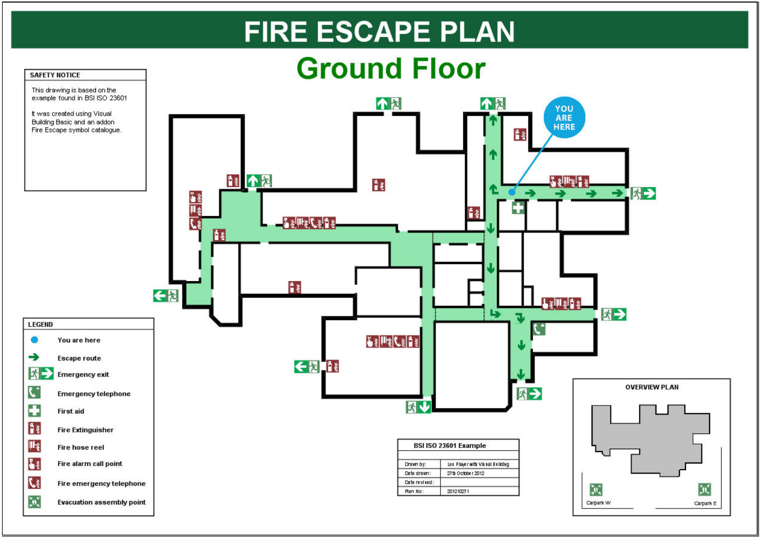
Transfer as much cheese as possible from Gruyere to Lausanne



Capacity: at most 20 cheeses can be transferred from Neuchatel to Lausanne

- a graph to model flow through edges (pipes)
- each edge has a capacity an upper bound on the flow rate (pipes have different sizes)
- Want to maximize rate of flow from the source to the sink

Tons of applications



FIRE ESCAPE PLAN

Ground Floor

SAFETY NOTICE

This drawing is based on the example found in BSI ISO 23601

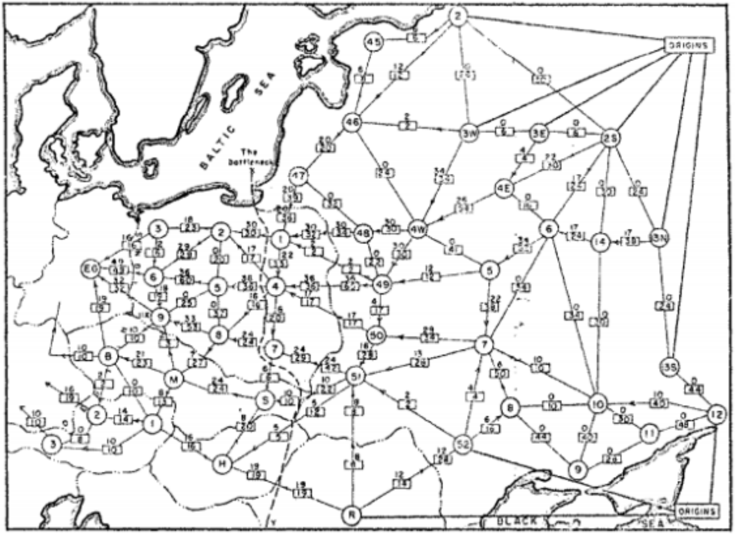
It was created using Visual Building Basic and an addon Fire Escape symbol catalogue.

LEGEND

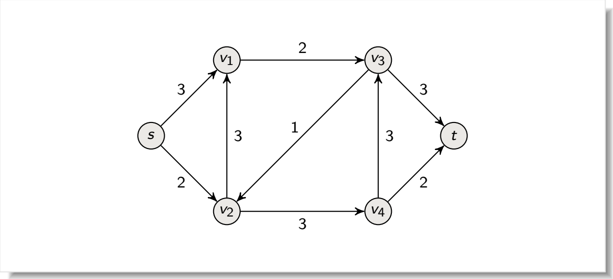
- ● You are here
- → Escape route
- Emergency exit
- ☎ Emergency telephone
- ✚ First aid
- Fire Extinguisher
- Fire hose reel
- Fire alarm call point
- ☎ Fire emergency telephone
- Evacuation assembly point

BSI ISO 23601 Example	
Drawn by:	Les Player with Visual Building
Date drawn:	27th October 2012
Date revised:	
Plan No:	201210271

Tons of applications

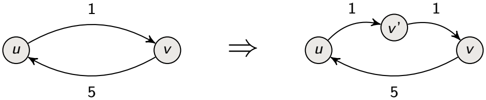


Flow Network (formally)



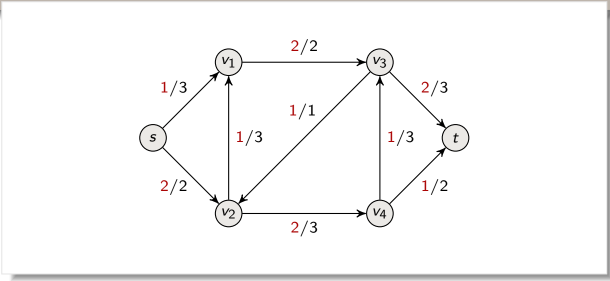
- Directed graph $G = (V, E)$
- Each edge (u, v) has a capacity $c(u, v) \geq 0$ $(c(u, v) = 0 \text{ if } (u, v) \notin E)$
- Source s and sink t (flow goes from s to t)
- No antiparallel edges (assumed w.l.o.g. for simplicity)

Why is “no antiparallel edges” w.l.o.g.?



- If there are two parallel edges (u, v) and (v, u), choose one of them say (u, v)
- Create a new vertex v'
- Replace (u, v) by two new edges (u, v') and (v', v) with $c(u, v') = c(v', u) = c(u, v)$
- Repeat this $O(E)$ times to get an equivalent flow network with no antiparallel edges.

Definition of a flow



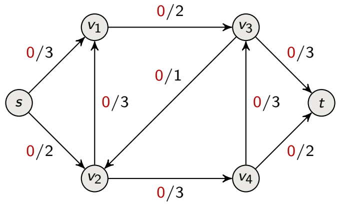
A flow is a function $f : V \times V \to \mathbb{R}$ satisfying:

Capacity constraint: For all $u, v \in V$: $0 \leq f(u, v) \leq c(u, v)$

Flow conservation: For all $u \in V \setminus \{s, t\}$,

$$\underbrace{\sum_{v \in V} f(v, u)}_{\text{flow into } u} = \underbrace{\sum_{v \in V} f(u, v)}_{\text{flow out of } u}$$

Definition of a flow



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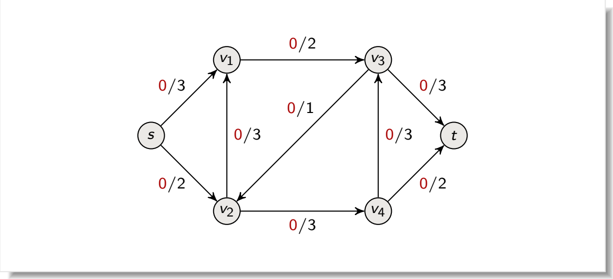
Value of a flow

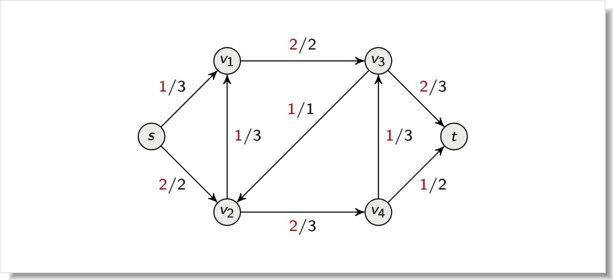


Value of a flow $f = |f|$

$$= \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s)$$

$=$ flow out of source $-$ flow into source

Value of a flow

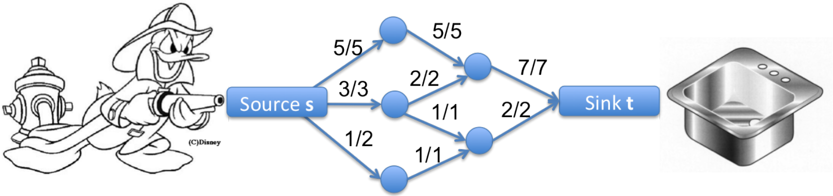


Value of a flow $f = |f|$

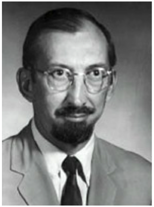
$$= \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s)$$

$$= \text{flow out of source} - \text{flow into source}$$

What’s the value of this flow? 9



L. R. Ford, Jr. (1927-)



D, R, Fulkerson (1924-1976)

MAXIMUM-FLOW PROBLEM

Ford-Fulkerson Method

The Ford-Fulkerson Method'54

```
FORD-FULKERSON-METHOD($G, s, t$):

1. Initialize flow $f$ to 0
2. while exists an augmenting path $p$ in the residual network $G_f$
3.         augment flow $f$ along $p$
4. return $f$
```

Basic idea:

- As long as there is a path from source to sink, with available capacity on all edges in the path
- send flow along one of these paths and then we find another path and so on

Applying the basic idea to examples

- As long as there is a path from source to sink, with available capacity on all edges in the path
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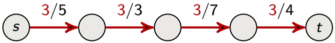
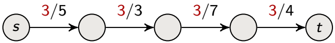


Exists a path **p** from s to t with remaining capacity
$\Rightarrow$ Push flow on **p**

Applying the basic idea to examples

- As long as there is a path from source to sink, with available capacity on all edges in the path
- send flow along one of these paths and then we find another path and so on



No path from s to t
with remaining capacity

and the flow is maximum

Applying the basic idea to examples

- As long as there is a path from source to sink, with available capacity on all edges in the path
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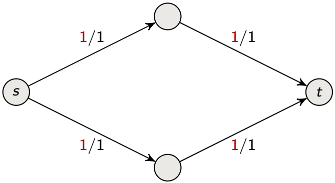

No path from s to t
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Applying the basic idea to examples

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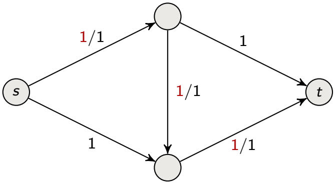

No path from *s* to *t*
with remaining capacity

but the flow is **not maximum**

What went wrong? How can we fix it?

Residual network

- Given a flow f and a network $G = (V, E)$
- the residual network consists of edges with capacities that represent how we can change the flow on the edges

Residual capacity:

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \quad \leftarrow \boxed{\text{Amount of capacity left}} \\ f(v,u) & \text{if } (v,u) \in E \quad \leftarrow \boxed{\text{Amount of flow that can be reversed}} \\ 0 & \text{otherwise} \end{cases}$$

Residual network:

$G_f = (V, E_f)$ where $E_f = \{(u,v) \in V \times V : c_f(u,v) > 0\}$

The Ford-Fulkerson Method'54

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1. Initialize flow $f$ to 0
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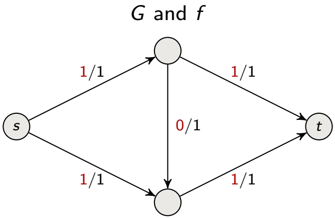
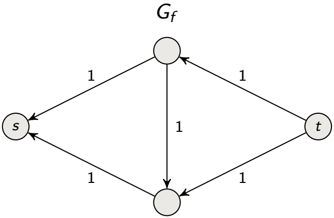
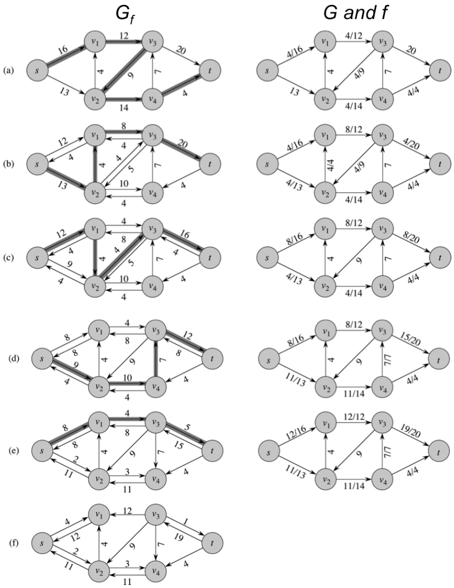
```

No augmenting path and flow of value 2 is optimal





Study and understand Example!


G_f

G and f

Summary

- Graphs fundamental object to study
- Two natural ways of traversing a graph: breadth-first search and depth-first search
- Topological sort of acyclic graphs by applying DFS and then order according to decreasing finishing times
- Strongly connected components
- Flow Networks